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# Topologies of Hyperspaces ( 巾空間のトポロジー )

筑波大学・数理解析科学研究科 数学専攻

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## 1 Introduction

In this note, we survey recent results on hypespaces with the Wijsman topology and the Attouch-Wets topology.

For a metric space  $X = (X, d)$ , let  $\text{Cld}(X)$  be the hyperspace of non-empty closed sets. By  $\text{Fin}(X)$ ,  $\text{Comp}(X)$  and  $\text{Bdd}(X)$ , we denote the subspaces of  $\text{Cld}(X)$  consisting of finite sets, compact sets and bounded closed sets, respectively. Let  $C(X)$  be the set of all continuous real-valued functions on  $X$ . By identifying each  $A \in \text{Cld}(X)$  with the map

$$X \ni x \mapsto d(x, A) = \inf_{a \in A} d(x, a) \in \mathbb{R},$$

we can regard  $\text{Cld}(X) \subset C(X)$ , whence  $\text{Cld}(X)$  has various topologies inherited from  $C(X)$ . The *Hausdorff metric topology* on  $\text{Cld}(X)$  is the topology of uniform convergence, the *Attouch-Wets topology* on  $\text{Cld}(X)$  is the topology of uniform convergence on bounded sets, and the *Wijsman topology* on  $\text{Cld}(X)$  is the topology of point-wise convergence, which depend on the metric  $d$  for  $X$ .

It should be remarked that the Attouch-Wets topology and the Wijsman topology are equal to the *Fell topology* on  $\text{Cld}(X)$  if  $X$  is a finite-dimensional normed linear space (cf. [2, p.142 & p.144]).

## 2 The Wijsman Topology

When we consider hyperspaces with the Wijsman topology, we denote  $\text{Cld}_W(X)$ ,  $\text{Fin}_W(X)$ ,  $\text{Bdd}_W(X)$ , etc. It is well-known that  $\text{Cld}_W(X)$  is metrizable if and only if  $X$  is separable, whence we can define an

admissible metric  $d_W$  for  $\text{Cld}_W(X)$  by using a countable dense set  $\{x_i \mid i \in \mathbb{N}\}$  in  $X$  as follows:

$$d_W(A, B) = \sup_{i \in \mathbb{N}} \min\{2^{-i}, |d(x_i, A) - d(x_i, B)|\}.$$

In [4], the following theorem is proved:

**Theorem 2.1.** *If  $X$  is an infinite-dimensional separable Banach space, then  $\text{Cld}_W(X)$  is homeomorphic to  $(\approx)$  the separable Hilbert space  $\ell_2$ .*

Also, for  $\text{Fin}_W(X)$  and  $\text{Bdd}_W(X)$ , similar results are proved in [4]:

**Theorem 2.2.** *If  $X$  is an infinite-dimensional separable Banach space, then*

$$\text{Fin}_W(X) \approx \text{Bdd}_W(X) \approx \ell_2 \times \ell_2^f,$$

where  $\ell_2^f = \{(x_i)_{i \in \mathbb{N}} \in \ell_2 \mid x_i = 0 \text{ except for finitely many } i \in \mathbb{N}\}$ .

To prove Theorems 2.1 and 2.2, we need characterizations of  $\ell_2$  and  $\ell_2 \times \ell_2^f$ . The following characterization of  $\ell_2$  is due to Toruńczyk [7] (cf. [8]):

**Theorem 2.3.** *In order that  $X \approx \ell_2$ , it is necessary and sufficient that  $X$  is a separable completely metrizable AR which has the discrete approximation property, that is,*

*Given a map  $f : \bigoplus_{n \in \mathbb{N}} \mathbb{I}^n \rightarrow X$ , there exist maps  $g : \bigoplus_{n \in \mathbb{N}} \mathbb{I}^n \rightarrow X$  arbitrarily close to  $f$  such that  $\{g(\mathbb{I}^n) \mid n \in \mathbb{N}\}$  is discrete in  $X$ .*

□

To state the characterization of  $\ell_2 \times \ell_2^f$  due to Bestvina and Mogilski [3], we need some notions. A metrizable space  $X$  is  $\sigma$ -completely metrizable if  $X$  is a countable union of completely metrizable closed subsets. A closed set  $A \subset X$  is a (strong)  $Z$ -set in  $X$  if there are maps  $f : X \rightarrow X \setminus A$  arbitrarily close to  $\text{id}$  (such that  $A \cap \text{cl } f(X) = \emptyset$ ). A countable union of (strong)  $Z$ -sets is called a (strong)  $Z_\sigma$ -set. When  $X$  itself is a (strong)  $Z_\sigma$ -set in  $X$ , we call  $X$  a (strong)  $Z_\sigma$ -space. For a class  $\mathcal{C}$  of spaces,  $X$  is *strongly universal* for  $\mathcal{C}$  if the following condition is satisfied:

Given a map  $f : A \rightarrow X$  of  $A \in \mathcal{C}$  such that  $f|B$  is a  $Z$ -embedding of a closed set  $B \subset A$ , there exist  $Z$ -embeddings  $g : A \rightarrow X$  arbitrarily close to  $f$  such that  $g|B = f|B$ .

In these definitions, the phrase ‘*arbitrarily close*’ is understood with respect to the *limitation topology*. In case  $X = (X, d)$  is a metric space, given a collection  $\mathcal{M}$  of maps from a space  $Y$  to  $X$ , a map  $f : Y \rightarrow X$  is *arbitrarily close* to maps in  $\mathcal{M}$  if for each  $\alpha : X \rightarrow (0, 1)$  there is  $g \in \mathcal{M}$  such that  $d(f(y), g(y)) < \alpha(f(y))$  for every  $y \in Y$ . The following is Corollary 6.3 in [3].

**Theorem 2.4.** *In order that  $X \approx \ell_2 \times \ell_2^f$ , it is necessary and sufficient that  $X$  is a separable  $\sigma$ -completely metrizable AR which is a strong  $Z_\sigma$ -space and is strongly universal for separable completely metrizable spaces.*

### 3 The Attouch-Wets Topology

When we consider hyperspaces with the Attouch-Wets topology, we denote  $\text{Cld}_{AW}(X)$ ,  $\text{Fin}_{AW}(X)$ ,  $\text{Bdd}_{AW}(X)$ , etc. Without the separability of  $X$ ,  $\text{Cld}_{AW}(X)$  is always metrizable and has an admissible metric  $d_{AW}$  defined as follows:

$$d_{AW}(A, B) = \sup_{n \in \mathbb{N}} \min \left\{ 1/n, \sup_{x \in X_n} \{|d(x, A) - d(x, B)|\} \right\},$$

where  $x_0 \in X$  is fixed and  $X_r = \{x \in X \mid d(x_0, x) \leq r\}$  for each  $r \in \mathbb{R}$ .

In [1], Banach, Kurihara and Sakai showed the following theorem:

**Theorem 3.1.** *If  $X$  is an infinite-dimensional Banach space with weight  $\tau$ , then  $\text{Cld}_{AW}(X) \approx \ell_2(2^\tau)$ , where  $\ell_2(\gamma)$  is the Hilbert space with weight  $\gamma$ .*

In [6], we have a following result which is analogous to Theorem 2.2:

**Theorem 3.2.** *For every infinite-dimensional Banach space  $X$  with weight  $\tau$ ,*

$$\begin{aligned} \text{Fin}_{AW}(X) &\approx \text{Comp}_{AW}(X) \approx \ell_2(\tau) \times \ell_2^f \quad \text{and} \\ \text{Bdd}_{AW}(X) &\approx \ell_2(2^\tau) \times \ell_2^f. \end{aligned}$$

Theorem 3.2 is based on the following theorem, which is obtained in [5] as the non-separable version of Bestvina-Mogilski's characterization.

**Theorem 3.3.** *In order that  $X \approx \ell_2(\tau) \times \ell_2^f$ , it is necessary and sufficient that  $X$  is a  $\sigma$ -completely metrizable AR with weight  $\tau$  which is a strong  $Z_\sigma$ -space and is strongly universal for  $\mathfrak{M}_1(\tau)$ , where  $\mathfrak{M}_1(\tau)$  is the space of all completely metrizable spaces with weight  $\tau$ .*

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